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An effective proof of the p-curvature conjecture for order one linear differential equations

joint work with Florian Fürnsinn.

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EThéN school in number theory.

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Power series hierarchy

$$y(x) = \sum_{n \geq 0} u_n x^n \in \mathbb{Q}[\![x]\!]$$

Rational

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D-finite series

$y(x)$ is D-finite if $\exists a_0(x), \ldots, a_r(x) \in \mathbb{Z}[x]$ not all zero such that $a_r(x)y^{(r)}(x) + \cdots + a_0(x)y(x) = 0.$

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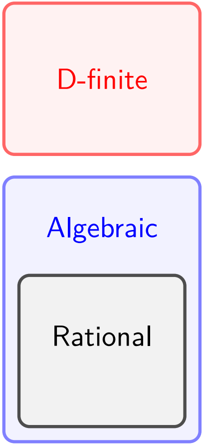
$\rightarrow y(x) = \exp(x^2+1)$ satisfies $y'(x) - 2xy(x) = 0$.



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Theorem (Abel, 1827)

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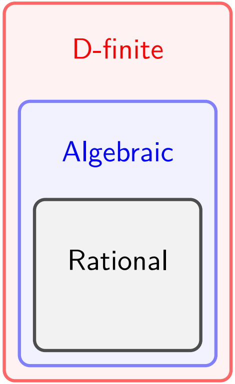

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Let $u(x) \in \overline{\mathbb{Q}(x)}$, decide if the nonzero solutions of $y'(x) = u(x)y(x)$ are algebraic.

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Special case of Grothendieck's conjecture, [Chudnovsky^2, 1985]

All solutions of $y'(x) = u(x)y(x)$ are algebraic over $\mathbb{Q}(x)$ *if and only if* for almost all prime numbers p, all solutions of $y'(x) = (u(x) \bmod p)y(x)$ are algebraic over $\mathbb{F}_p(x)$.

The p-curvature conjecture

$$\begin{cases} y'(x) &= u(x)y(x) \qquad \color{red}{(\mathrm{Eq})} \\ y'(x) &= (u(x) \bmod p)y(x) \qquad \color{red}{(\mathrm{Eq})_p} \end{cases} \quad \text{with } u(x) = \frac{a(x)}{b(x)} \in \mathbb{Q}(x).$$

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To (Eq)$_p$, attach the p-curvature $u^{(p-1)}(x) + u(x)^p \bmod p \in \mathbb{F}_p(x)$ [Jacobson, 1937].

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The p-curvature is zero if and only if $(\mathrm{Eq})_p$ has a basis of algebraic solutions.

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Special case of Grothendieck's p-curvature conjecture, [Chudnovsky^2, 1985; Honda, 1974]

All solutions of (Eq) are algebraic over $\mathbb{Q}(x)$ *if and only if* for almost all prime numbers p, the p-curvature of $(\mathrm{Eq})_p$ vanishes.

From p-curvatures to polynomials

Theorem (Honda, 1974)

The p-curvature conjecture holds for equations $y'(x) = u(x)y(x)$ *with* $u(x) = \frac{a(x)}{b(x)}$, $a(x), b(x) \in \overline{\mathbb{Q}}[x]$.

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Theorem (Kronecker, 1880; Chebotarev, 1926)

Let $R(w) \in \mathbb{Q}[w]$ *be irreducible. If for almost all prime numbers* p *the polynomial* $R(w) \bmod p$ *has a root in* $\mathbb{F}_p$*, then* $R(w)$ *has a root in* $\mathbb{Q}$*, and hence it is linear.*

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Spoiler: Honda's Theorem is equivalent to Kronecker's Theorem.

Towards effectivity

Theorem (Rothstein, 1976; Trager, 1976)

Let $u(x) \in \mathbb{Q}(x)$ be a rational function of the form

$$u(x) = \frac{a(x)}{b(x)} = \sum_{i=1}^{r} \frac{\alpha_i}{x - \beta_i},$$

with $a(x), b(x) \in \mathbb{Z}[x]$. Then the residues α_i are precisely the roots of

$$R(w) := \mathrm{res}_x(b(x), a(x) - w \cdot b'(x)) \in \mathbb{Z}[w].$$

Proof of Honda's Theorem

$$y'(x) = u(x)y(x) \text{ (Eq)} \quad \text{with } u(x) = \frac{a(x)}{b(x)}, \text{ and } R(w) := \mathrm{res}_x(b(x), a(x) - w \cdot b'(x)).$$

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Proposition (Folklore; Honda, 1981)

The following are equivalent:

(1) (Eq) *has an algebraic solution.*

(2) *We have* $\deg a(x) < \deg b(x)$*,*
all poles of $u(x)$ *are simple, and*
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Proposition (Honda, 1981)

Let p be a prime number. TFAE:

(1)$_p$ (Eq)$_p$ *has an algebraic solution in* $\mathbb{F}_p[\![x]\!]$.

(2)$_p$ *We have* $\deg a(x) < \deg b(x)$*, all poles of* $u(x)$ *are simple, and* $R(w)$ *splits completely in* $\mathbb{F}_p[w]$.

(3)$_p$ *We have* $u(x)^p + u^{(p-1)}(x) \bmod p = 0$.

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Kronecker's Theorem: $(2)_p$ for almost all prime numbers p implies (2).

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Kronecker's Theorem: $(2)_p$ for almost all prime numbers p implies (2).

Can we deduce (2) from $(2)_p$ for a *finite* number of primes?

Effective Kronecker

Theorem (Chudnovsky^2, 1985)

Let $R(w) \in \mathbb{Z}[w]$ with leading coefficient $\Delta \in \mathbb{Z}$.
There exists $\sigma \in \mathbb{N}$ such that $R(w)$ splits completely over $\mathbb{Q}$ if and only if $R(w) \bmod p$ splits completely over $\mathbb{F}_p$ for all primes p:

- *not dividing Δ,*
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Theorem (Fürnsinn-P., 2025+)

In the previous theorem, one can choose $\sigma = (2M+1)N + 2M$ *with* $M := \lceil 2.826 \cdot \Delta^3 \cdot t(\Delta) \rceil$, $N := \lceil 6.076BM \rceil$, *where* $t(\Delta) := \prod_{p|\Delta} p^{1/(p-1)}$ *and* $B \in \mathbb{R}$ *is an upper bound on the modulus of all complex roots of* $R(w)$.

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Criterion: If $p \le \sigma$, $p \nmid \Delta$ and $R(w) \bmod p$ does not split completely in $\mathbb{F}_p$, then $R(w)$ does not split completely in $\mathbb{Q}$.

Hermite-Padé approximation

Given power series $f_1(x), \ldots, f_r(x) \in \mathbb{Q}[\![x]\!]$ and $n, s \in \mathbb{N}$, find polynomials $P_i(x) \in \mathbb{Q}[x]$ such that $\deg(P_i(x)) \leq n$ and

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[Hermite, 1873] e is transcendental, [Padé, 1894], [Mahler, 1931].

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Idea to prove algebraicity: With $f_i(x) = f^{i-1}(x)$, $f(x)$ is algebraic *if and only if* for the optimal P_i's, the remainder $P_1(x) + P_2(x)f(x) + \cdots + P_r(x)f^{r-1}(x)$ vanishes for large n, r.

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$$P_0(z) + P_1(z)(1-z)^\alpha + \cdots + P_{2M}(z)(1-z)^{2M\alpha} = gz^\sigma + O(z^{\sigma+1})$$

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with $\sigma = (2M+1)N + 2M$, $g = \dfrac{N!^{2M+1}}{\sigma!} \in \mathbb{Q}^*$, $P_0(0) = \left(\displaystyle\prod_{j=1}^{2M} \binom{j\alpha+N-1}{N}\right)^{-1}$.

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For all $\gamma \in L \setminus \{0\}$, $\underbrace{\left|\mathrm{den}(\gamma)^{[L:\mathbb{Q}]}\,\mathrm{Norm}_{L/\mathbb{Q}}(\gamma)\right|}_{\in\mathbb{Z}} \geq 1.$

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For all $\gamma \in L \setminus \{0\}$, $\underbrace{\left|\operatorname{den}(\gamma)^{[L:\mathbb{Q}]} \operatorname{Norm}_{L/\mathbb{Q}}(\gamma)\right|}_{\in\mathbb{Z}} \geq 1$.

Construct $\gamma_{M,N} \in L$, $\gamma_{M,N} \neq 0$, such that when $N >> M >> 0$,

$$\left|\operatorname{den}(\gamma_{M,N})^{[L:\mathbb{Q}]} \operatorname{Norm}_{L/\mathbb{Q}}(\gamma_{M,N})\right| < 1.$$

Effective Honda

Corollary [Chudnovsky², 1985; Fürnsinn-P., 2025+]

Let $a(x), b(x) \in \mathbb{Z}[x]$, $\deg(a(x)) < n := \deg(b(x))$ and
$R(w) := \mathrm{res}_x(b(x), a(x) - w \cdot b'(x)) \in \mathbb{Q}[w]$,
with leading coefficient $\Delta := \mathrm{res}_x(b(x), -b'(x))$, $t := \prod_{p|\Delta} p^{1/(p-1)}$.
Let $B \in \mathbb{R}$ be an upper bound on the modulus of all complex roots of $R(w)$.
Let $M := \lceil 2.826 \cdot \Delta^3 \cdot t(\Delta) \rceil$ and $N := \lceil 6.076 BM \rceil$.

All solutions of $y'(x) = \frac{a(x)}{b(x)} y(x)$ are algebraic *if and only if* the p-curvatures of the differential equation vanish for all primes p:

- not dividing Δ;
- at most $\sigma := (2M+1)N + 2M$.

New algorithm and complexity

Input $a(x), b(x) \in \mathbb{Z}[x]$, $b(x)$ squarefree, $\deg(a(x)) < \deg(b(x))$.

Output The nature (algebraic or transcendental) of the solutions of $y'(x) = \frac{a(x)}{b(x)} y(x)$.

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1. $R(w) := \mathrm{res}_x(b(x), a(x) - w \cdot b'(x)) \in \mathbb{Q}[w]$, Δ, t, B;
2. $M := \lceil 2.826 \cdot \Delta^3 \cdot t(\Delta) \rceil$, $N := 10BM$, $\sigma := (2M+1)N + 2M$, $p \leftarrow 2$;
3. **while** $p \leq \sigma$:
   1. **if** $p \nmid \Delta$, **then** compute the p-curvature;
   2. **if** p-curvature $\neq 0$, **then** return transcendental, **else** $p \leftarrow \mathrm{nextprime}(p)$;
4. return algebraic.

New algorithm and complexity

Input $a(x), b(x) \in \mathbb{Z}[x]$, $b(x)$ squarefree, $\deg(a(x)) < \deg(b(x)) = n$. Coefficients bounded by H.

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- Computing p-curvatures, [Bostan-Schost, 2009]

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1. $R(w) := \mathrm{res}_x(b(x), a(x) - w \cdot b'(x)) \in \mathbb{Q}[w]$, Δ, t, B; $\tilde{O}(n^2 \log(H))$ bit operations
2. $M := \lceil 2.826 \cdot \Delta^3 \cdot t(\Delta) \rceil$, $N := 10BM$, $\sigma := (2M+1)N + 2M$, $p \leftarrow 2$;
3. **while** $p \leq \sigma$: $\tilde{O}(n^2 \sigma)$ bit operations
   i. **if** $p \not| \Delta$, **then** compute the p-curvature;
   ii. **if** p-curvature $\neq 0$, **then** return transcendental, **else** $p \leftarrow \mathrm{nextprime}(p)$;
4. return algebraic.

- Computing p-curvatures, [Bostan-Schost, 2009]
- $\sigma = \tilde{O}((Hn)^{12n})$.

New algorithm and complexity

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Comparison

Alternative approaches: Given $a(x), b(x) \in \mathbb{Z}[x]$ of degree at most n and coefficients at most H, one can either compute $R(w) = \mathrm{res}_x(b(x), a(x) - w \cdot b'(x))$ and find its rational roots, or compute the indicial equations to decide transcendence.

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Degree	Height	p-curv	IE	RT+RR
10	2^{10}	1 ms	12 ms	3 ms
20	2^{10}	2 ms	24 ms	10 ms
20	2^{20}	2 ms	25 ms	21 ms
160	2^{10}	0.4 s	1.8 s	2.4 s
160	2^{20}	0.4 s	1.9 s	4.0 s

Table: Average computation time of algorithms deciding transcendence of solutions on random rational function inputs of prescribed degree and height.

Perspectives

Make all proved cases of Grothendieck's p-curvature conjecture effective.

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Thank you for your attention.